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OSCILLATION FREQUENCY EVOLUTION OF A DISSIPATIVE SYSTEM†

M. B. SEVRYUK and A. P. SEIRANYAN

Moscow

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The evolution of the oscillation frequencies of linear autonomous systems with potential and dissipative forces depending on a one-dimensional parameter is discussed. Over- and underdamping cases are considered. It is shown that as the parameter is changed the oscillation frequencies of a dissipative system in many cases behave in the complex plane like frequencies of several decoupled oscillators with one degree of freedom.

AN EXTENSIVE literature [1–9] is devoted to the stability of dissipative systems. Below we investigate the evolution of the complete set of system eigenvalues in the entire complex plane (and not just the transition of eigenvalues across the imaginary axis, which is responsible for the onset or loss of stability) and then consider cases in turn where the potential energy matrix, the dissipative forces matrix and the mass matrix are varied.

1. The equations of motion of a linear mechanical system with dissipative and potential forces have the form

$$M\ddot{q}+D\dot{q}+Aq=0 \tag{1.1}$$

where M, A and D are real symmetric square matrices of order n which respectively express the kinetic energy, potential energy and dissipative function, and q is the generalized coordinate vector of dimension n.

We consider the matrix A in the form

$$A=C-pB \tag{1.2}$$

where $p\geqslant 0$ is the load parameter, and C and B are real symmetric matrices. The dependence of the form (1.2) on the load parameter characterizes so-called simple systems [4]. It is assumed that the matrices M, D, C and B are positive-definite. The condition $D>0$ means that system (1.1) is purely dissipative.

We shall investigate the dependence on p of the eigenvalues and the stability of system (1.1). After making the substitution $q=Xe^{\lambda t}$ we arrive at the generalized eigenvalue problem

$$L(\lambda,p)X=0,\quad L(\lambda,p)=\lambda^2M+\lambda D+C-pB \tag{1.3}$$

where X is an eigenvector (EVec) of dimension n and λ is its eigenvalue (EVal).

There are $2n$ EVals λ_i and corresponding EVecs X_i associated with problem (1.3). The

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EVals λ are governed by the characteristic equation

$$\det L(\lambda, p) = 0 \tag{1.4}$$

The matrices M, D, C and B and the parameter p are real, so that alongside λ and X the complex conjugate quantities $\bar{\lambda}$ and $\bar{X}$ are also an EVal and EVec, respectively for the problem

$$L(\bar{\lambda}, p)\bar{X} = 0 \tag{1.5}$$

The matrix operators in (1.3) and (1.5) are mutually conjugate.

If the λ_i are simple roots of Eq. (1.4) the general solution of the equations of motion (1.1) has the form

$$q(t) = \sum_{i=1}^{2n} \alpha_i X_i \exp(\lambda_i t) \tag{1.6}$$

where α_i are constants determined from the initial conditions. The expansion (1.6) is also valid for roots λ_i of multiplicity r if the number of linearly independent EVecs X_i corresponding to λ_i is also r.

Suppose the EVal λ in (1.3) corresponds to the EVec X. We multiply the first equality in (1.3) scalarly by X and consider the result as a quadratic equation in λ Its solutions are equal to

$$\lambda_{1,2} = (-d \pm \sqrt{S})/(2m), \quad S = d^2 - 4m(c - pb) \tag{1.7}$$

$$d = (DX, X), m = (MX, X), \quad c = (CX, X), \quad b = (BX, X) \tag{1.8}$$

$$(X, Y) = X^T \bar{Y} = \sum_{j=1}^{n} x_j \bar{y}_j$$

We normalize the vector X

$$(X, X) = 1 \tag{1.9}$$

Since the matrices M, D, C and B are positive-definite, we find from (1.8) that

$$m > 0, d > 0, c > 0, b > 0 \tag{1.10}$$

If the EVal λ is real, the corresponding EVec X is also real, apart from an arbitrary complex multiplier. For simplicity we will assume it to be real.

We consider the discriminant S in (1.7). Suppose λ is the EVal corresponding to the EVec X. If $S < 0$, then in view of (1.10), the roots (1.7) will be complex-conjugate. Hence both roots in (1.7) are EVals of problem (1.3) with corresponding complex-conjugate EVecs. However, if $S > 0$, the roots (1.7) are real, and only one of them is an EVal λ of problem (1.3) with corresponding EVec X. The second root of (1.7) may be unnecessary. One cannot establish a priori the correct sign in front of the root in (1.7).

According to (1.7) and (1.10), when $p = 0$ all the EVals lie in the left complex half-plane $\operatorname{Re}\lambda < 0$. System (1.1) is therefore stable. As p increases some of the λ may move into the right half-plane, which means that stability is lost.

By doubling the dimensions the generalized EVal problem (1.3) can be reduced to the ordinary problem $KU = \lambda U$ with non-symmetric matrix $K(p)$. A generic one-parameter family of real matrices is characterized [10, 11] by simple EVals, and at isolated values of the parameter by double real EVals λ_0 with a Jordan block of order 2. More complicated singularities can be avoided by an arbitrarily small perturbation of the family. We consider the coalescence of the complex-conjugate pair λ and $\bar{\lambda}$ into one real EVal λ_0. With the coalescence of the

EVals the complex-conjugate EVecs X and $\bar{X}$ corresponding to λ and $\bar{\lambda}$ also merge together. The case when a single EVec X_0 corresponds to a double root λ_0 is called a strong interaction [12],† characterized by perpendicular approach and separation directions for EVals in the complex plane.

At the point of multiplicity $p = p_0$ the EVals lose differentiability. For small $\Delta p = p - p_0$ the expansions for λ have the form [12]

$$\lambda = \lambda_0 \pm \lambda_1 \sqrt{\Delta p} + O(|\Delta p|)$$
$$\lambda_1^2 = (BX_0, X_0)/[(L_1(\lambda_0)X_1, X_0) + (MX_0, X_0)]$$
$$L_1(\lambda_0) = \partial L(\lambda_0, p_0)/\partial\lambda = 2\lambda_0 M + D$$
$$L(\lambda_0, p_0)X_0 = 0, \ L(\lambda_0, p_0)X_1 = -L_1(\lambda_0)X_0$$

(the adjoint vector X_1 being defined apart from the summarized κX_0).

Because λ_0 is real, the vectors X_0 and X_1 can also be chosen to be real. The numerator in the expression for λ_1^2 is positive, and generically the denominator does not vanish [12], i.e. the strong interaction is not degenerate.

In the present case the orthogonality condition for strong EVal interaction [12] takes the form

$$(L_1(\lambda_0)X_0, X_0) = 0 \tag{1.11}$$

Using this to express λ_0, we conclude that $\lambda_0 < 0$ in view of the positive definiteness of D and M. Hence for dissipative systems strong interaction on the real axis is only possible for negative λ_0. Having reached the real axis, simple EVals λ cannot leave it because otherwise this would mean the appearance of additional roots $\bar{\lambda}$. Thus strong interaction is a mechanism for ensuring that complex-conjugate EVal pairs reach the axis and the departure of an EVal pair from the real axis. At $p = p_0$ in the general solution $q(t)$ of the equations of motion (1.1) the secular terms

$$\alpha_1 X_0 \exp(\lambda_0 t) + \alpha_2 (X_0 t + X_1)\exp(\lambda_0 t)$$

appear, where a_1 and a_2 are constants. This contradicts the assertion [7, pp. 91, 95] that the general solution for the dissipative system (1.1) is always of the form (1.6).

We will compute the derivative of the simple EVal λ with respect to the parameter p. Suppose the EVec X corresponds to this EVal. Using the fact that $(\bar{\lambda}, \bar{X})$ is the solution of the conjugate problem (1.5), we obtain [13] by the perturbation method

$$d\lambda/dp = b_c/(2\lambda m_c + d_c) \tag{1.12}$$

$$b_c = (BX, \bar{X}), \ m_c = (MX, \bar{X}), d_c = (DX, \bar{X}) \tag{1.13}$$

If the vector X is complex, the quantities b_c, m_c and d_c are also in general complex. If the vector X is real, then according to (1.8) and (1.13) $b_c = b$, $m_c = m$, $d_c = d$. Using (1.7) we obtain in this case

$$d\lambda/dp = \pm bS^{-1/2} \tag{1.14}$$

Below we analyse the behaviour of the EVals in the complex plane when the parameter p varies.

†See also: SEIRANYAN A. P., Interaction of eigenvalues. Preprint No. 446, Inst. Probl. Mekh. Akad. Nauk SSSR, Moscow, 1990.

2. We shall first consider the overdamping case when $S>0$ for all non-zero X for some value of $p \leqslant 0$. According to (1.7) here all the EVals are real. Moreover, any EVal λ is obtained from (1.7) either with a plus sign for all EVecs X corresponding to this λ, or with a minus sign for all EVecs X [3, Theorem 7.4]. We call an EVal λ of the first type primary, and of the second type secondary. Exactly n primary and n secondary EVals exist when taking multiplicities into account [3, Theorem 7.6], and every primary EVal is greater than every secondary EVal [3, Theorem 7.7]. In the overdamping case, the matrix $L(\lambda, p)$ (1.3) is semi-simple [3, 12], i.e. to every EVal of multiplicity r there correspond r linearly independent EVecs X.

Remark. The theorems of Sec. 7.6 in [3], referring to the case of overdamping in the oscillatory system (1.1), are proved with the additional condition $A>0$. However, all these theorems, apart from Theorem 7.2, remain true for any symmetric matrix A. The key condition turns out to be the positivity of the discriminant $S=d^2-4ma$, where $a=(AX, X)$.

The inequality $S>0$ is satisfied for all non-zero X and for all $p \geqslant 0$ if

$$(\mu^D_{\min})^2 - 4\mu^M_{\max}\,\mu^C_{\max} > 0 \tag{2.1}$$

where $\mu_{\min}$, $\mu_{\max}$ are the smallest and largest eigenvalues of the matrices D, M and C. Indeed, using relations (1.9) and (1.10) we find

$$S=d^2-4mc+4pmb \geqslant d^2-4mc \geqslant$$
$$\geqslant (\mu^D_{\min})^2 - 4\mu^M_{\max}\,\mu^C_{\max} > 0$$

We call condition (2.1) the sufficient condition for overdamping. From now on in this section we shall assume that this condition is satisfied. Then according to (1.7), when $p=0$ all the EVals λ are negative, while for $p>0$ they are at least real because $S>0$.

According to (1.10) and (1.14), for primary simple EVals λ' and secondary simple EVals λ' we have

$$d\lambda'/dp = bS^{-1/2} > 0,\quad d\lambda''/dp = -bS^{-1/2} < 0 \tag{2.2}$$

Because the number of primary λ' and secondary λ'' EVals is equal to n, the picture of EVal dependence on p is as follows: when $p=0$ all the λ are negative, and as the parameter increases the n primary λ' increase monotonically while the n secondary λ'' decrease monotonically. In Fig. 1(a) the EVals are shown by circles, and the arrows show the direction of motion of λ as p increases.

When two EVals collide they pass through one another without leaving the real axis, and to a double root λ_0 there correspond two linearly independent EVecs. This type of EVal coalescence, characterized by unchanged direction of their motion as they approach and separate, is called weak interaction [12] (see also the footnote on p. 603). During the weak interaction the EVals remain differentiable at the moment of collision.

During their rightward motion along the real axis the primary EVals λ' can pass through zero. Putting $\lambda=0$ in (1.3), we obtain the problem

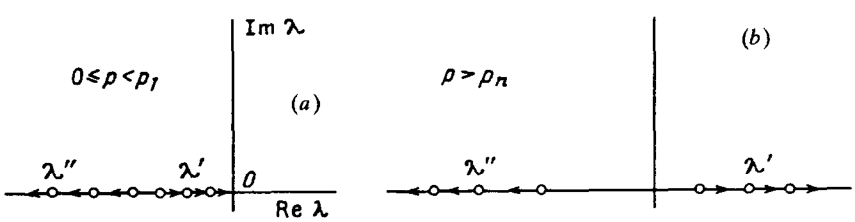

FIG. 1.

$$CX = pBX \tag{2.3}$$

Thus the passage of λ' through zero is observed at values of the parameter p equal to the EVals $p_1, p_2, \ldots, p_n$ of problem (2.3). Here $c = pb$ and, according to (1.7) and (2.2)

$$\lambda' = 0, \; d\lambda'/dp = b/d > 0$$

The smallest EVal $p_1 = p_c$ of problem (2.3) is critical: when $p > p_c$ system (1.1) loses stability statically (divergence). When $p < p_c$ the motion of system (1.1) is monotonically damped because all the λ are negative.

When $p > p_n$, where p_n is the largest EVal of problem (2.3), all n primary EVals λ' of the original problem (1.3) become positive. When p is increased further all the λ' increase without limit, and the n secondary EVals λ'' decrease without limit (Fig. 1b). This agrees with earlier results [5].

The behaviour of the EVals λ is somewhat more complicated in the case of multiply-loaded p, i.e. when for some EVal p_k of problem (2.3) there correspond $g > 1$ linearly independent EVecs $X_1, X_2, \ldots, X_g$. This means that to the parameter value $p = p_k$ there corresponds a g-fold EVal $\lambda = 0$ with EVecs $X_1, X_2, \ldots, X_g$ in the original problem. The expansion of a multiple zero for small $|\Delta p|$, $\Delta p = p - p_k$, has the form $\lambda = \lambda_1 \Delta p + o(|\Delta p|)$, where the quantities λ_1 are found from the equation [12]

$$\det \| (BX_i, X_j) - \lambda_1 (DX_i, X_j) \| = 0, i, j = 1, 2, \ldots, g$$

In view of the positive definiteness of matrices B and D all g roots λ_1 of this equation are positive. Consequently, as p increases beyond p_k, all the g primary EVals λ' cross from the negative to the positive semi-axis simultaneously.

Such a case is, for example, realized when the matrices B and C are identical: $B = C$. Then when $p = p_c = 1$ the EVal problem (1.3) acquires the form $(\lambda^2 M + \lambda D)X = 0$. This implies that n negative EVals are governed by the problem $DX = -\lambda'' MX$, while the remaining n EVals are zero: $\lambda' = 0$. With increasing $p > 1$ these n EVals λ' cross to the positive semi-axis and separate.

3. To understand the EVal evolution in the case of a negative discriminant S we will first consider a special case. Suppose $D = \gamma M$, where $\gamma = \text{const} > 0$. Substituting this expression into (1.3), we arrive at the problem of the oscillations of a conservative system

$$(-M\omega^2 + C - pB)X = 0 \quad (-\omega^2 = \lambda^2 + \gamma\lambda) \tag{3.1}$$

From this we have $\omega^2 = (c - pb)/m$. When $p = 0$ all n EVals ω_i^2 of problem (3.1) are positive. We assume that $\omega_1^2 \leqslant \omega_2^2 \leqslant \ldots \leqslant \omega_n^2$. According to Rayleigh's theorem all the frequencies ω^2 decrease as p increases

$$d\omega^2/dp = -b/m < 0$$

(This formula was obtained by the perturbation method [13] using the fact that ω^2 and X are real in (3.1).) When $p = p_1, p_2, \ldots, p_n$, where the p_k are given by (2.3), the EVals ω^2 pass through zero and they all become negative as p increases further.

We express λ in terms of ω^2 and find the derivative with respect to p

$$\lambda = -\gamma/2 \pm \sqrt{S}, \; d\lambda/dp = \pm b/(2m\sqrt{S})$$
$$S = \gamma^2/4 - \omega^2 = \gamma^2/4 + (pb - c)/m \tag{3.2}$$

If $S < 0$, the plus and minus signs in (3.2) correspond to complex-conjugate EVals λ.

We shall assume that the constant γ satisfies the inequality $\gamma^2/4<\omega_1^2$. Then when $p=0$ we have $S<0$ for all EVecs X because $c/m\leqslant\omega_1^2$ for any non-zero X. Consequently, according to (3.2) all the λ are complex at $p=0$, with $\operatorname{Re}\lambda=-\gamma/2<0$. This means that all the EVals λ lie in the left complex half-plane along a line parallel to the imaginary axis and displaced from it by the distance $\gamma/2$ (Fig. 2a). As p increases, the complex-conjugate λ approach each other along this line and merge pairwise (when $S=0$), and then separate along the real axis in different directions (Fig. 2b). The EVals λ pass through zero when $p=p_1,\ p_2,\ldots,\ p_n$.

The inequality $\gamma^2/4>\omega_n^2$ is the strong overdamping condition. This case was considered in Sec. 2. If however $\omega_1^2<\gamma^2/4<\omega_n^2$, then for $p=0$ some of the EVals are complex and some are negative. According to (3.2), when p increases the complex-conjugate λ approach one another along the line $\operatorname{Re}\lambda=-\gamma/2$, merge pairwise (when $S=0$), and then separate in different directions along the real axis. As far as the negative λ are concerned, half of them (λ') move to the right and the other half (λ'') to the left along the real axis, and for all λ', λ'' and all $p\leqslant 0$ we have $\lambda''<-\gamma/2<\lambda'$.

4. We will now consider the case of underdamping, when at $p=0$ the discriminant $S\ll 0$ for all X satisfying (1.9). Here, according to (1.7), all the EVals λ are complex quantities that are close to the frequencies of the corresponding conservative system. Such a situation is found, for example, when the inequality

$$S=d^2-4mc\leqslant(\mu_{\max}^D)^2-4\mu_{\min}^M\mu_{\min}^C\ll 0 \tag{4.1}$$

is satisfied, where $\mu_{\max}$, $\mu_{\min}$ are the largest and smallest eigenvalues of the matrices D, M and C.

To investigate this case we introduce a small damping ϵD, where $\epsilon>0$ is a small parameter, D is a positive definite matrix, and in accordance with (4.1) $\mu_{\max}^D\sim(\mu_{\min}^M\mu_{\min}^C)^{1/2}$. Problem (1.3) is written in the form

$$L_2(\lambda,p,\epsilon)X=0,\ L_2(\lambda,p,\epsilon)=\lambda^2M+\epsilon\lambda D+C-pB \tag{4.2}$$

Assuming λ to be a simple EVal, we expand λ and X in terms of ϵ

$$\lambda=\lambda_0+\epsilon\lambda_1+\ldots,\qquad X=X_0+\epsilon X_1+\ldots \tag{4.3}$$

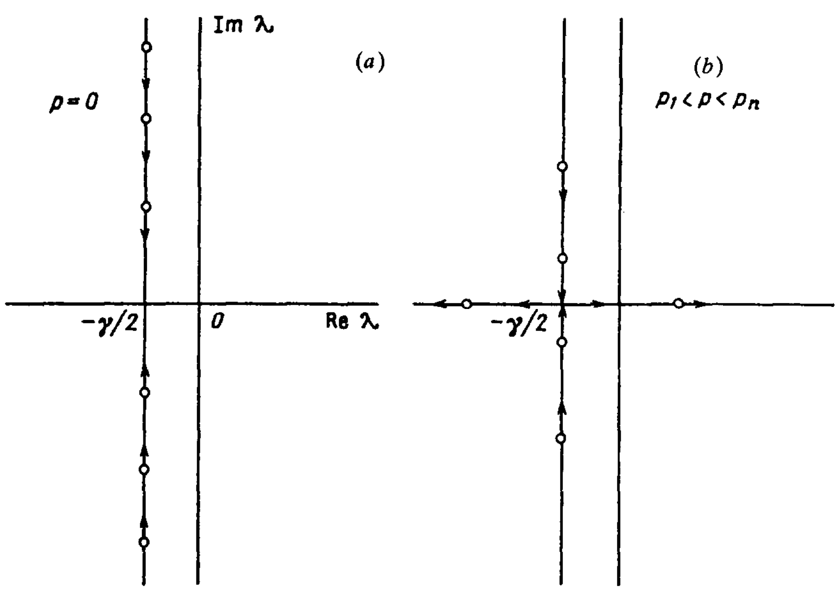

FIG. 2.

Substitutuing (4.3) into (4.2), for the first terms in the expansions we find

$$L_2(\lambda_0, p, 0)X_0 = 0 \tag{4.4}$$

$$\lambda_1 = -\tfrac{1}{2}(DX_0, X_0)/(MX_0, X_0) \tag{4.5}$$

$$L_2(\lambda_0, p, 0)X_1 = -\lambda_0(2\lambda_1 M + D)X_0$$

(this formula for λ_1 is obtained by the perturbation method [13] using the fact that λ_0^2 and X_0 are real in (4.4)). For $\epsilon\lambda_1$ one can obtain the limits

$$-\epsilon\nu_{max}/2 \leqslant \epsilon\lambda_1 \leqslant -\epsilon\nu_{min}/2 \tag{4.6}$$

where $\nu_{max} > 0$ and $\nu_{min} > 0$ are the largest and smallest EVals of the problem $DX = \nu MX$.

The behaviour of the frequencies of the conservative system corresponding to the null approximation (4.4) has already been discussed in Section 3. According to (4.5) and (4.6) the presence of weak damping shifts all the frequencies of the conservative system by $\epsilon\lambda_1 < 0$ in the first approximation.

We expand the derivative $d\lambda/dp$ with respect to ϵ. Because λ and X are, in general, complex, we use expression (1.12). Substituting expansion (4.3) into (1.12), to a first approximation in ϵ we obtain

$$\begin{aligned} d\lambda/dp &= \tfrac{1}{2}b_0(\lambda_0 m_0)^{-1} + \epsilon k \\ k &= (\lambda_0 m_0)^{-1}\,[b_1 - \tfrac{1}{2}b_0(\lambda_0 m_0)^{-1}(\lambda_1 m_0 + 2\lambda_0 m_1 + d_0/2)] \\ b_0 &= (BX_0, \bar{X}_0),\ m_0 = (MX_0, \bar{X}_0), d_0 = (DX_0, \bar{X}_0) \\ b_1 &= (BX_0, \bar{X}_1),\ m_1 = (MX_0, \bar{X}_1) \end{aligned} \tag{4.7}$$

For small p all the quantities λ_0 are purely imaginary, the λ_1 are real, the X_0 can be chosen to be real, while the X_1, according to (4.5), can be chosen to be purely imaginary. With such a choice of X_0 and X_1 the quantities b_0, m_0 and d_0 are positive, while b_1, and m_1 are purely imaginary. The coefficient k is therefore real, while the quantity $b_0(2\lambda_0 m_0)^{-1}$ is purely imaginary, and if $\mathrm{Im}\lambda_0 > 0$, then $\mathrm{Im}(d\lambda/dp) < 0$, and conversely if $\mathrm{Im}\lambda_0 < 0$, then $\mathrm{Im}(d\lambda/dp) > 0$ (Fig. 3). Thus for purely imaginary λ_0 the derivative $d\lambda/dp$ is a complex number whose real part is of order ϵ. If however λ_0 is real, then all the quantities in (4.5)–(4.7) are real.

The evolution of the oscillation frequencies of an underdamped system is illustrated in Fig. 3. For $p=0$ the complex EVals λ lie in the strip (4.6). As p increases they approach one another, remaining within the limits of this strip, merge pairwise and then separate along the real axis in opposite directions. The transition of an EVal through zero, as before, is only governed by the matrix $A = C - pB$. The system loses stability when $p > p_1$.

5. We will investigate the evolution of the EVal λ as the parameter changes in the case of arbitrary damping $D > 0$.

When $p > p_n$, where p_n is the largest EVal of problem (2.3), the discriminant S is positive for all non-zero X. Thus when $p > p_n$, system (1.1), (1.2) becomes strongly overdamped. According to the results of Section 2, n positive primary EVals λ' move to the right along the real axis as $p > p_n$ increases, while n negative secondary EVals λ'' move along it to the left.

Suppose that when $p=0$ all the λ are complex quantities, which according to (1.7) is equivalent to the condition that the discriminant S is negative at $p=0$ for all EVecs X. We will describe the evolution of the EVal λ as p decreases from large positive values $p > p_n$ to zero. Initially, when the parameter decreases, the n positive primary EVals λ' decrease monotonically, while the n negative secondary λ'' increase monotonically. When $p > p_n$ the primary EVals meet the secondary ones. When λ' and λ'' collide, we have $S=0$, which according to (1.7) can be written in the form $2\lambda m + d = 0$. The latter equality is the orthogonality condition (1.11), meaning the strong interaction of λ' and λ'' [12] (see also footnote on p. 603). Thus as p decreases there are n strong pair interactions, as a result of

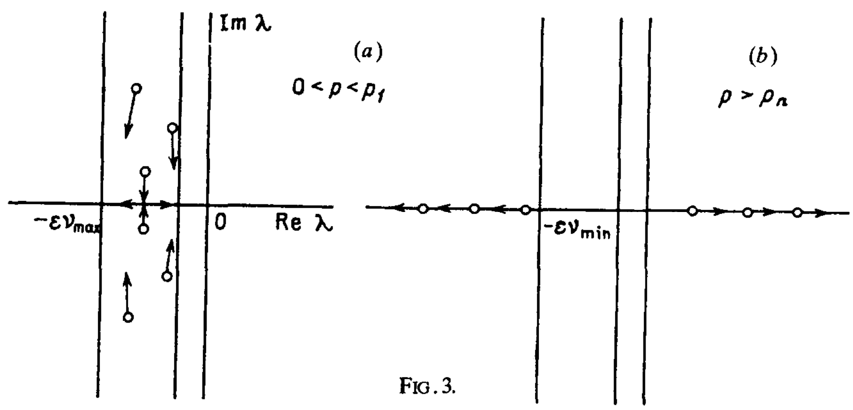

FIG. 3.

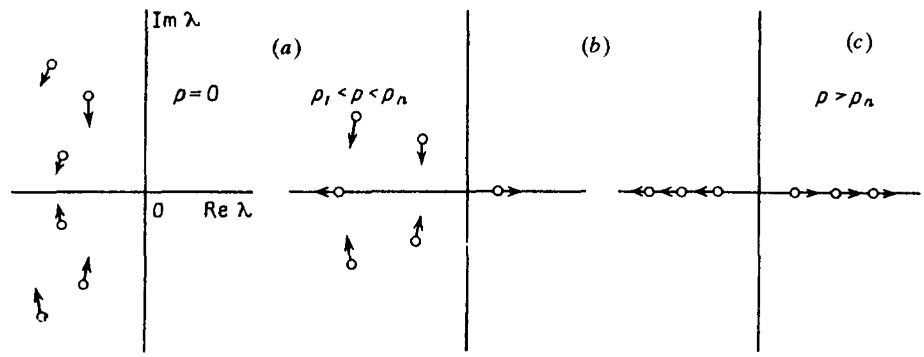

FIG. 4.

which the real λ become complex conjugate quantities. The evolution of the EVals is shown in Fig. 4, the arrows showing the motion of the λ as p increases.

If, however, not all the λ are complex at $p=0$, then as p decreases from large positive values to zero some of the EVal pairs approach but do not meet and remain on the real axis. For these λ the discriminant S remains positive. The evolution of EVals in this case begins from the position shown in Fig. 4(b) (the EVal configuration for large p corresponds to Fig. 4(c) as before).

6. Above we considered simple systems characterized by a linear dependence of the matrix A on the load parameter p [4]. However, the results obtained can also be generalized to the non-linear case, so long as the matrix $A(p)$ is, as before, symmetric for all $p \geqslant p_0$, the matrix $A(p_0)$ is positive definite, and the matrix $B=B(p)=dA/dp$ is negative definite for $p \geqslant p_0$, where p_0 is a fixed number. The difference from the case previously considered lies in the total number of EVals crossing on to the positive semi-axis as p increases. Their number is given by the number of EVals for the problem $A(p)X=0$ when $p \geqslant p_0$, and may be less than n. The critical values p_k are found from the equation det $A(p)=0$. If det $A(p) \neq 0$ for all $p \geqslant p_0$, then all the λ remain in the left half-plane as p increases.

7. Consider problem (1.1) when the dissipative force matrix $D(p)$ is varied. We will assume that $D(p_0)>0$ and $D_1=dD/dp<0$ when $p \geqslant p_0$. We will also assume that the constant matrices M and A are positive definite. In the previous notation (1.8) and (1.13) we have

$$\lambda=\frac{-d \pm \sqrt{S}}{2m}, \quad \frac{d\lambda}{dp}=-\frac{\lambda d_{c1}}{2\lambda m_c+d_c} \qquad (7.1)$$
$$S=d^2-4ma, \quad a=(AX, X), \quad d_{c1}=(D_1X, \bar{X})$$

Quantities with the subscript c are complex if X is a complex vector. For real $X \neq 0$ we have $m_c = m > 0$, $d_c = d$, $d_{c1} = d_1 = (D_1 X, X) < 0$ in view of $M > 0$, $D_1 = dD/dp < 0$.

We shall first consider the case of fairly heavy damping $D(p_0)$ so that when $p = p_0$ we have the inequality $S > 0$ for all non-zero X (the overdamping condition). Since when $p = p_0$ we have $d > 0$, from (7.1) and using $m > 0$, $a > 0$ it follows that all the EVals λ are negative. As all the corresponding EVecs X are real, we shall drop the subscript c in the second equality in (7.1) and, substituting the first equality in (7.1) into the second, we obtain

$$d\lambda/dp = -\tfrac{1}{2} d_1 m^{-1} (1 \mp dS^{-\frac{1}{2}}) \tag{7.2}$$

From the inequality $S > 0$ for all non-zero X it follows that when $p = p_0$ there are n primary EVals λ' and n secondary λ'', with $\lambda'' < \lambda'$ for all λ' and λ''. The plus sign in (7.1) and the minus sign in (7.2) correspond to the primary λ', and the minus sign in (7.1) and the plus sign in (7.2) correspond to the secondary λ''. From (7.2), using $d_1 < 0$, $m > 0$, $a > 0$, $d > 0$, $S > 0$, it follows that the derivatives of the primary EVals λ' with respect to p are negative, and those of the secondary λ'' are positive.

Consequently, as $p > p_0$ increases the n EVal pairs move to meet one another, and strong interaction occurs when they meet $(S = 0)$, with the EVals becoming complex-conjugate. The evolution of one EVal pair is shown in Fig. 5.

When p is increased further the λ, $\bar{\lambda}$ pair may pass into the right half-plane. System (1.1) then loses oscillation stability (flutter occurs). At the critical point $p = p_c$ we have $\lambda = \pm i\omega$ and $d = 0$. It is interesting to note that it is not necessary for $D(p_c)X = 0$ to hold, and the vector $d\lambda/dp\,|_{p_c}$ can have any direction.

Example. We put

$$n = 2,\ M = \mathrm{diag}(3, 1),\ A = \mathrm{diag}(2, 2)$$

$$D = D(p) = \begin{Vmatrix} -(1+2x)\cdot p & 1+yp \\ 1+yp & -p \end{Vmatrix}$$

where x and y are small. The matrices M and A are positive definite. When $x = y = 0$ the matrix $D_1 = dD/dp$ (independent of p) is negative definite, while when $p = p_0 = -6$ the matrix $D(p_0)$ is positive definite and the overdamping condition

$$(\mu_{\min}^D)^2 - 4\mu_{\max}^M \mu_{\max}^A = 1 > 0$$

is satisfied (cf. (2.1)). These properties of $D(p)$ also hold for small non-zero x and y.

The characteristic equation for EVals λ when $p = 0$ as the simple purely imaginary root $\lambda = i$ with corresponding EVec

$$\sqrt{2}\,X = \begin{Vmatrix} i \\ 1 \end{Vmatrix}$$

and $d\lambda/dp\,|_{p=0} = x + yi$. Thus

$$\sqrt{2}\,D(0)\,X = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \begin{Vmatrix} i \\ 1 \end{Vmatrix} = \begin{Vmatrix} 1 \\ i \end{Vmatrix} \neq 0$$

(and, moreover, the matrix $D(0)$ is non-degenerate), while $d\lambda/dp$ can have real and imaginary parts of either sign depending on the choice of x and y. (Here we digress from the question of whether $p = 0$ is the smallest value of the parameter p at which the EVals pass through the imaginary axis.)

This example shows that the behaviour of the system frequencies when the matrix of the dissipative forces D varies can be very complicated, and certainly more complicated than when $n = 1$.

Figure 5 shows the full possible evolution of any pair of EVals λ when the positive definite matrix $D(p)$ satisfying the overdamping condition at $p = p_0$ becomes negative definite when p increases, the absolute values of its eigenvalues increasing without limit. If these conditions are not satisfied, the EVal pair undergoes part of the journey presented in Fig. 5. For example, if the matrix $D(p)$ remains positive definite for $p \geqslant p_0$, the λ do not cross over into the right half-plane.

The set of evolutions of the n EVal pairs shown in Fig. 5 determines the evolution of the complete collection of EVals of system (1.1) for monotonic $(dD/dp < 0)$ variation of the dissipative forces matrix $D(p)$.

8. Consider problem (1.1) when the mass matrix $M(p)$ varies. We shall assume that the initial mass $M(p_0)$ is fairly small for the overdamping condition ($d^2 > 4ma$ for all non-zero X) to be satisfied at $p = p_0$. Assuming that $M_1 = dM/dp > 0$ for $p \geqslant p_0$, we will investigate the effect of increasing mass on the oscillation frequencies. The constant symmetric matrices D and A are assumed to be positive definite.

The EVals λ are computed from the first relation in (7.1), and if λ is real, then

$$\frac{d\lambda}{dp} = \frac{dm_1}{2m^2}\left[1 \mp \frac{1 - 2amd^{-2}}{(1 - 4amd^{-2})^{1/2}}\right], \quad m_1 = (M_1 X, X) \tag{8.1}$$

In the case of very small mass $(m \ll d^2/(4a))$ in the leading approximation in m we obtain

$$\lambda' = -\frac{a}{d} - \frac{a^2 m}{d^3}, \quad \lambda'' = -\frac{d}{m} + \frac{a}{d} \qquad \frac{d\lambda'}{dp} = -\frac{a^2 m_1}{d^3}, \quad \frac{d\lambda''}{dp} = \frac{dm_1}{m^2} \tag{8.2}$$

Because of the positivity of the quantities m, a, d and m_1 it again follows from (8.1) and (8.2) that as p increases n pairs of negative λ', λ'' approach one another, coalesce in pairs, and then separate at right angles to the real axis (strong interaction). A qualitative picture of the evolution of one pair of EVals is shown in Fig. 6. When the mass increases without limit (when the smallest eigenvalue of the matrix M increases) all the frequencies tend to zero, and $\mathrm{Im}\lambda/\mathrm{Re}\lambda \to \infty$. The general picture of the evolution of the set of EVals is given by the totality of separate interacting pairs shown in Fig. 6.

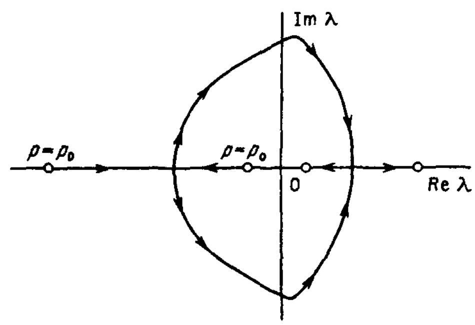

FIG. 5.

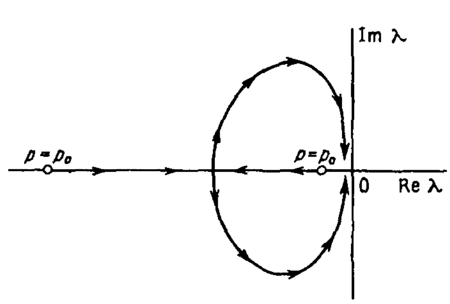

FIG. 6.

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